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Stochastic Particle Acceleration and Statistical Closures[1]

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In a recent paper, Maasjost and Elsässer (ME) concluded, from the results of numerical experiments and heuristic arguments, that the Bourret and the direct-interaction approximation (DIA) are "of no use in connection with the stochastic acceleration problem" because (1) their predictions were equivalent to that of the simpler Fokker–Planck (FP) theory, and (2) either all or none of the closures were in good agreement with the data. Here some analytically tractable cases are studied and used to test the accuracy of these closures. The cause of the discrepancy (2) is found to be the highly non-Gaussian nature of the force used by ME, a point not stressed by them. For the case where the force is a position-independent Ornstein–Uhlenbeck (i.e., Gaussian) process, an effective Kubo number K can be defined. For $K \ll 1$ an FP description is adequate, and conclusion (1) of ME follows; however, for $K \gtrsim 1$ the DIA behaves much better qualitatively than the other two closures. For the non-Gaussian stochastic force used by ME, all common approximations fail, in agreement with (2).

KEY WORDS: Stochastic acceleration, closure approximations, Fokker–Planck theory, Bourret approximation, direct-interaction approximation.

1. INTRODUCTION

The problem of acceleration of charged particles in a prescribed stochastic electromagnetic field has received attention both for its importance as a basic phenomenon in laboratory plasma physics and plasma

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astrophysics[1,2] and as a model system on which to test various statistical closure schemes.[3,4] In the present paper, we discuss some results on the latter topic.

The problem can be cast into the following mathematical form. Find the evolution of the ensemble-averaged Green's function $G(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t')$ which satisfies

$$G(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t) = \langle \tilde{G}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t') \rangle \tag{1}$$

where $\tilde{G}$ satisfies the Liouville equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \mathbf{b}(\mathbf{x}, \mathbf{v}, t) \cdot \frac{\partial}{\partial \mathbf{v}}\right) \tilde{G}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t') = 0 \qquad (t \geqslant t') \tag{2a}$$

subject to the initial condition

$$\tilde{G}(\mathbf{x}, \mathbf{v}, t'; \mathbf{x}', \mathbf{v}', t') = \delta(\mathbf{x} - \mathbf{x}')\, \delta(\mathbf{v} - \mathbf{v}') \tag{2b}$$

In eq. (1), "$\langle \dots \rangle$" denotes an ensemble average over realizations of $\mathbf{b}$. Here $\mathbf{b}(\mathbf{x}, \mathbf{v}, t)$ is a prescribed random acceleration field specified by the set of its many-argument moments $\langle \mathbf{b}(\mathbf{x}, \mathbf{v}, t) \rangle$, $\langle \mathbf{b}(\mathbf{x}, \mathbf{v}, t)\, \mathbf{b}(\mathbf{x}', \mathbf{v}', t') \rangle$, and so on. Now consider the ensemble-averaged distribution function $f(\mathbf{x}, \mathbf{v}, t) \doteq \langle \tilde{f}(\mathbf{x}, \mathbf{v}, t) \rangle$, where $\tilde{f}$ satisfies (2a) together with any initial condition of the form

$$\tilde{f}(\mathbf{x}, \mathbf{v}, t_0) = f_0(\mathbf{x}, \mathbf{v}) \tag{3}$$

and where $f_0(\mathbf{x}, \mathbf{v})$ is statistically independent of $\mathbf{b}(\mathbf{x}, \mathbf{v}, t)$. Given G, f can then be found from

$$f(\mathbf{x}, \mathbf{v}, t) = \int d\mathbf{x}'\, d\mathbf{v}'\, G(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t_0)\, f_0(\mathbf{x}', \mathbf{v}') \tag{4}$$

In the following we assume that $\mathbf{b}$ is stationary, homogeneous, and independent of $\mathbf{v}$. If we are interested only in the evolution of the velocity distribution

$$h(\mathbf{v}, t) \doteq \int d\mathbf{x}\, f(\mathbf{x}, \mathbf{v}, t) \tag{5}$$

then upon integrating eqs. (4) and (1) with respect to $\mathbf{x}$ we obtain

$$h(\mathbf{v}, t) = \int d\mathbf{v}'\, P(\mathbf{v}, t - t_0; \mathbf{v}')\, h_0(\mathbf{v}') \tag{6}$$

where

$$h_0(\mathbf{v}) \doteq \int \mathbf{dx}\, f_0(\mathbf{x}, \mathbf{v}) \tag{7a}$$

and

$$P(\mathbf{v}, t-t_0; \mathbf{v}') \doteq \int \mathbf{d}(\mathbf{x}-\mathbf{x}')\, G(\mathbf{x}-\mathbf{x}', \mathbf{v}, t-t_0; \mathbf{v}') \tag{7b}$$

We have used the homogeneity and stationarity assumptions to simplify the argument dependences on $\mathbf{x}$ and t.

Despite the fact that the acceleration field $\mathbf{b}$ is externally specified, it is not, in general, possible to find a closed set of equations for any finite set of ensemble averages of products of powers of $\mathbf{b}$ and nonzero powers of $\tilde{G}$. Some approximation scheme is usually necessary in order to obtain such closed, deterministic equations. Thompson and Hubbard,[5] Sturrock,[1] and Hall and Sturrock[2] used the Fokker–Planck theory to obtain a velocity space diffusion equation for h in which the diffusion coefficient is expressed in terms of the covariance of the fluctuating force. (In a related calculation, Hubbard[6] also calculated the field fluctuations for a near-equilibrium plasma and included polarization effects in the drag coefficient, thereby obtaining the Balescu–Lenard equation. Such calculations include the important effects of self-consistency, which is, however, beyond the scope of this article.) This theory has been formulated more generally in terms of cumulant expansions by Kubo[7] and others for linear differential equations with stochastic (operator) coefficients. (Bourret,[8] van Kampen,[9] and Keller[10] have explicitly worked out the same scheme to higher orders.) Truncated cumulant expansions such as the Fokker–Planck theory are valid if and only if (a) the rms value of an appropriate norm of the stochastic coefficient multiplied by its autocorrelation time (the Kubo number) is much less than 1 (Ref. 11; the "narrowing condition"), and (b) the coefficient operator is near-Gaussian in the sense that the time integral over all but one of the time variables of its nth cumulant for each $n \geq 3$ is negligible compared to the corresponding integral for $n=2$. Bourret gave an integral equation for h valid under the same conditions. More generally, Orszag and Kraichnan[3] treated the problem in the direct-interaction approximation (DIA). Allegedly, that theory should have some relevance for Kubo numbers larger than 1. From the assumptions made in the derivation of the DIA and the fact that it reduces to the Fokker–Planck theory in the small Kubo number limit, it is clear that the DIA only applies when a Gaussian condition, which reduces to that for the Fokker–Planck theory in the small Kubo number limit, holds. We return to this point later.

The present work was motivated by a recent paper of Maasjost and Elsässer (ME),[12] in which they presented a test of the Fokker–Planck, the Bourret, and the direct-interaction approximation as applied to the stochastic acceleration problem. Their method consisted of (1) calculating the phase space Green's function from the results of a numerical simulation, (2) inserting the result into the expressions forming the left- and right-hand sides of the equations resulting from the closures in question (integrated over a small velocity interval), and (3) observing whether or not the left- and right-hand sides agreed. They found that, "depending on the parameter regime, either all or none of the three theories are good models for the stochastic acceleration problem," and concluded that, in particular, the DIA is "of no use in connection with the stochastic acceleration problem." This conclusion is striking, since it has been argued that the DIA should remain reasonable, if not quantitatively precise, as the nonlinearity becomes large, whereas the Fokker–Planck and Bourret approximations become ill-behaved in that limit.[13,14] Furthermore, the disagreement that they found persists for times much longer than the effective autocorrelation time.

These surprising results have motivated us to further discuss the same problem, using, however, mainly analytical solutions that can be obtained in tractable cases. We show that the conclusions of Maasjost and Elsässer are intimately bound to the fact that the force field that they used in their numerical simulations is highly non-Gaussian, and are not, in general, correct for Gaussian fields. For example, for a stationary Gaussian force field, not without practical interest, we find a parameter regime in which the DIA behaves much better qualitatively than the other two closures.

The procedure of ME, which we follow in this paper, is clearly not a sufficient test of a statistical closure. In fact, its necessity can also be questioned, since it does not solve the closure self-consistently. We discuss this issue further in Section 6. In this paper we take the point of view that the work of ME certainly provides some information about the closures in question, and that our work represents an attempt to clarify their conclusions.

The remainder of the paper is organized as follows. In the main body of the paper we consider the special case where the acceleration is position-independent. In Section 2, we give the formal solution to the stochastic acceleration problem in that special case, and relate that solution to the stochastic oscillator. We briefly discuss in Section 3 the basic properties of the stochastic acceleration fields and give an explicit evaluation of the Green's function for each of the three closures mentioned previously. There, we also define an effective Kubo number for the stochastic acceleration problem. A summary of the equations resulting from the three closures is given in Section 4 along with a comment on a heuristic argument given by ME. In Section 5, we display and discuss the results of inserting our analytically obtained Green's functions into the closure equations of Section 4. We present our conclusions in Section 6. Appendix A contains some extensions of the results of Sections 2, 3, and 5 for cases where the stochastic acceleration is Gaussian, varies spatially as well as temporally, and has a finite correlation length l_c. Some properties of random fields of the type used by ME are given in Appendix B.

2. FORMAL SOLUTION AND THE EXACTLY SOLVABLE CASE

For simplicity, we specialize in this paper to the one-dimensional case. For higher spatial dimensionality, the formal manipulations generalize trivially. Again, we consider only statistically stationary and homogeneous fields $b(x, t)$.

2.1. Solution

Upon solving eq. (2) for $\tilde{G}$ by integrating along the characteristics

$$\frac{dx(t)}{dt} = v(t) \tag{8a}$$

$$\frac{dv(t)}{dt} = b[x(t), t] \tag{8b}$$

performing the ensemble average in (1), and using the translational invariance of G with respect to x and x' to change the integration variable in (7) to x', we find

$$P(v, \tau; v') = \langle \delta[v_0(x, v, t+\tau; t \mid b) - v'] \rangle \tag{9}$$

where $v_0(x, v, t+\tau; t \mid b)$ is the velocity at time t of a particle that has position x and velocity v at time $t+\tau$ for a given realization b of the acceleration field.

In the remainder of the main body of this paper, we further specialize to the case when b is independent of x. Extensions to x-dependent acceleration fields are addressed in the appendices. (In particular, it is shown there that a simple modification of this theory suffices for the parameter values used by ME.) The integration of eq. (8b) then reduces to

$$v_0 = b - \int_0^t dt'\, b(t') \tag{10}$$

Then, upon inserting (10) into (9), using the Fourier representation of the δ function and exchanging the order of integration and ensemble averaging, we obtain the solution

$$P(v, \tau; v') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp[ik(v - v')] \left\langle \exp\left(-ik \int_0^{\tau} dt' \, b(t') \right) \right\rangle \tag{11}$$

The velocity space Green's function for a particle acted on by a position-independent acceleration field is formally the same as the configuration space Green's function for a point in a fluid with a position-independent random velocity field, a problem which has been treated by Kubo.[11] Equation (11) can be obtained from Kubo's eq. (5.8) by Fourier transforming the δ function.

2.2. Relationship to the Stochastic Oscillator

Here we consider stationary accelerations $b(t)$, not necessarily Gaussian, for which

$$\langle b(t) \rangle = 0 \tag{12a}$$

$$\langle b(t_1)\, b(t_2) \rangle = b_0^2 \exp(-|t_1 - t_2|/\tau_c) \tag{12b}$$

A normalized acceleration $\hat{b}$ can be defined by

$$\hat{b}(\eta) \doteq b(\tau_c \eta)/b_0 \tag{13}$$

and a stochastic oscillator Green's function $R_K(\eta)$ can be defined by

$$R_K(\eta) \doteq \langle \tilde{R}_K(\eta) \rangle$$

where

$$[(\partial/\partial\eta) + iK\hat{b}(\eta)]\, \tilde{R}_K(\eta) = \delta(\eta) \qquad [\tilde{R}_K(\eta) = 0 \text{ for } \eta < 0] \tag{15}$$

K is precisely the Kubo number for the system (15), and Kubo's "narrowing condition"[7] is $K \ll 1$. The solution to (15) is, formally

$$R_K(\eta) = \left\langle \exp\left[-iK \int_0^{\eta} d\eta' \, \hat{b}(\eta') \right] \right\rangle \tag{16}$$

Upon changing the integration variable in (11) to $K \doteq kb_0\tau_c$, we obtain

$$P(v, \tau; v') = (b_0\tau_c)^{-1} \int \frac{dK}{2\pi} \exp\left[iK \left(\frac{v - v'}{b_0 \tau_c} \right) \right] R_K(\tau/\tau_c) \tag{17}$$

Thus, the solution to the stochastic acceleration problem when the acceleration satisfies (12) is a Fourier transform with respect to the Kubo number of the solution of the stochastic oscillator.

3. GAUSSIAN AND MAASJOST–ELSÄSSER FIELDS, AND THE SOLUTIONS OF THE STOCHASTIC OSCILLATOR AND ACCELERATION PROBLEMS

3.1. Gaussian Acceleration Field

There are several ways in which a Gaussian field (or any other stochastic field) can be characterized.[9] Let $P_n(b_1, x_1, t_1;\ b_2, x_2, t_2; ...;\ b_n, x_n, t_n) \prod_{i=1}^{n} db_i$ be the probability that the n vector $(b_1, b_2, ..., b_n)$ lies in the volume element $\prod_{i=1}^{n} (b_i, b_i + db_i)$. If all of the multivariate distributions P_n are jointly Gaussian for all $n \geqslant 1$, then b is a Gaussian random field. Alternatively, $b(x, t)$ is a centered Gaussian if its characteristic functional has the form

$$\begin{aligned} G[k] &\doteq \left\langle \exp\left[-i \int_{-\infty}^{\infty} dx\, dt\, k(x, t)\, b(x, t) \right] \right\rangle \\ &= \exp\left[-\frac{1}{2} \int_{-\infty}^{\infty} dx_1\, dx_2\, dt_1\, dt_2\, k(x_1, t_1) \langle b(x_1, t_1)\, b(x_2, t_2) \rangle\, k(x_2, t_2) \right] \end{aligned} \tag{18}$$

If b is independent of x, then by comparing (16) with the definition (18) of $G[k]$ we see that $R_K(\tau)$ is a special case of $G[k]$ evaluated for

$$k(x', t') = \left(\frac{K}{b_0 \tau_c} \right) \chi_{(0,t)}(t')\, \delta\{ x' - [x - v(t - t')] \} \tag{19}$$

where

$$\begin{aligned} \chi_A(y) &\doteq 1 \qquad (y \in A) \\ &\doteq 0 \qquad (y \notin A) \end{aligned} \tag{20}$$

Upon inserting (19) into (18), we obtain for b an Ornstein–Uhlenbeck process

$$R_K(\tau) = \exp[-K^2 \alpha(\tau)] \tag{21}$$

where

$$\alpha(\tau) \doteq \frac{1}{2} \int_0^{\tau} d\tau_1\, d\tau_2 \langle \hat{b}(\tau_1)\, \hat{b}(\tau_2) \rangle = \tau + e^{-\tau} - 1 \tag{22}$$

We then obtain from (17)

$$P(v, \tau; v') = \{2b_0\tau_c[\pi\alpha(\tau/\tau_c)]^{-1/2}\} \exp\left(-\frac{1}{4\alpha(\tau/\tau_c)}\frac{(v-v')^2}{b_0^2\tau_c^2}\right) \quad (23)$$

3.2. Maasjost and Elsässer Field

The acceleration field used by Maasjost and Elsässer[12] is given in the continuum limit by

$$b(x, t) = b_x(x)\, b_t(t) \quad (24)$$

where b_x and b_t are mutually independent, stationary Gaussian processes satisfying

$$\langle b_x(x)\rangle = 0 \quad (25a)$$

$$\langle b_t(t)\rangle = 0 \quad (25b)$$

$$\langle b_x(x_1)\, b_x(x_2)\rangle = \sigma_x^2 \exp(-|x_1 - x_2|/l_c) \quad (25c)$$

$$\langle b_t(t_1)\, b_t(t_2)\rangle = \sigma_t^2 \exp(-|t_1 - t_2|/\tau_c) \quad (25d)$$

$$\sigma_x^2\sigma_t^2 \doteq b_0^2 \quad (25e)$$

The field b is clearly non-Gaussian since, for example, its one-point distribution function is given by

$$P_1(b) = (\pi b_0)^{-1}\, K_0(|b|/b_0) \quad (26)$$

The characteristic functional for b can be expressed formally as (see Appendix B)

$$G[k] = \exp\left(-\tfrac{1}{2}\,\mathrm{tr}\ln\left[\delta(x_1 - x_2) + b_0^2 \int dx_3\, dt_3\, dt_4\, k(x_3, t_3)\, k(x_2, t_4) \times \exp(-|x_1 - x_3|/l_c - |t_3 - t_4|/\tau_c)\right]\right) \quad (27)$$

where the quantity in the square brackets on the right-hand side is the kernel of the integral operator on which the operations outside are performed. Upon inserting (19) into (27) with $l_c = \infty$, we find

$$R_K(\tau) = [1 + 2K^2\alpha(\tau)]^{-1/2} \quad (28)$$

where $\alpha(\tau)$ is given in (22). Then from (17) we obtain

$$P(v, \tau; v') = \{\pi b_0\tau_c[2\alpha(\tau/\tau_c)]^{-1/2}\}\, K_0\left(\frac{|v - v'|}{b_0\tau_c[2\alpha(\tau/\tau_c)]^{1/2}}\right) \quad (29)$$

3.3. Effective Kubo Number

As stated in Section 2.1, the random velocity field problem studied by Kubo[11] is equivalent to the $l_c = \infty$ stochastic acceleration problem. According to Kubo, the narrowing condition for the stochastic acceleration problem is

$$\bar{K} \doteq b_0 \tau_c / \Delta v \ll 1 \tag{30}$$

where

$$\Delta v \doteq \|\partial/\partial v\|^{-1} \tag{31}$$

and where "$\|\partial/\partial v\|$" stands for some characteristic value of $\partial/\partial v$ applied to the resulting distribution function. If the velocity distribution is near-Gaussian, then Δv is a measure of the spread of the distribution. In general, Δv must be interpreted instead as in (31). For the Green's functions that we are studying, we can rewrite (30) in a form that generalizes easily to non-Gaussian velocity distributions, viz.

$$\bar{K}(\tau) \doteq \lim_{\rho \to \infty} \left(\frac{\int_0^\rho dK\, K^\gamma R_K(\tau)}{\int_0^\rho dK\, R_K(\tau)} \right)^{1/\gamma} \qquad (\gamma > 0) \tag{32}$$

For γ an even integer $[\bar{K}(\tau)]^\gamma$, if finite, is the magnitude of $\partial^\gamma/\partial v^\gamma$ acting on P at $v = v'$. Upon inserting (21) into (32), we obtain for the Gaussian process, to within a dimensionless constant factor,

$$\bar{K}(\tau) \approx [\alpha(\tau)]^{-1/2} \sim \tau^{-1/2} \qquad (\tau \gg 1) \tag{33}$$

For the ME field, the interpretation of Δv as a spread is inappropriate and (32) yields

$$\bar{K} = \infty \qquad (\forall \tau) \tag{34}$$

The definition of $\bar{K}$ here is a heuristic generalization of that for the stochastic oscillator. It will be used in the present work in the discussion of the plots displayed in Section 5, but not as an expansion parameter in any closure scheme.

4. STATISTICAL CLOSURES IN THE CONTEXT OF THE STOCHASTIC ACCELERATION PROBLEM

For a summary of the equations resulting from the Fokker–Planck, the Bourret, and the direct-interaction approximations, and for the description of the test method used by ME, we refer the reader to Ref. 12. For convenience, we merely write here those equations in the forms used by ME. These expressions, which represent the probability flux of particles leaving the interval (v_l, v_r), are, respectively

$$\frac{\partial}{\partial\tau}\int_{v_l}^{v_r} dv\, P(v, \tau; v') = b_0^2\left(\frac{1}{\tau_c}+\frac{v}{l_c}\right)^{-1}\frac{\partial}{\partial v}P(v, \tau; v')|_{v=v_l}^{v=v_r} \tag{35a}$$

$$\frac{\partial}{\partial\tau}\int_{v_l}^{v_r} dv\, P(v, \tau; v') = \int_0^{\tau} d\tau' \;\langle bb\rangle(v\tau', \tau')\frac{\partial}{\partial v}P(v, \tau-\tau'; v')|_{v=v_l}^{v=v_r} \tag{35b}$$

$$\frac{\partial}{\partial\tau}\int_{v_l}^{v_r} dv\, P(v, \tau; v') = \int_0^{\tau} d\tau'' \int_{-\infty}^{\infty} dv'' \int_{-\infty}^{\infty} dx''\, G(x'', v, \tau''; v'')\langle bb\rangle(x'', \tau'')$$
$$\times \frac{\partial}{\partial v}P(v'', \tau-\tau''; v')|_{v=v_l}^{v=v_r} \tag{35c}$$

We also note two points. First, it is clear that in the ME test the right-hand sides of the three closure schemes will agree with each other to within a relative error of order τ_{ac}/τ_{ev}, where τ_{ac} is the effective autocorrelation time for $v\in(v'-\Delta v, v'+\Delta v)$ and τ_{ev} is the characteristic time scale of the evolution of P. Over most of the time interval in each of the results displayed by ME, τ_{ac}/τ_{ev} is a small quantity. Thus ME appear to have made no effort to study parameter regimes in which the three closures would not be expected to agree with each other and where, therefore, the advantages of one over the other two might be displayed.

Second, ME, in their analytical discussion of closure schemes (in their Sect. 2) attempt to motivate a statement that the exact Green's function $\tilde{G}$ can be approximated by the ensemble-averaged Green's function G only if the latter is approximately equal to the unperturbed Green's function G_0. They do this by using the assumption (which they do not justify) that

$$\delta G \doteq \tilde{G}-G = -G(\delta L)G \tag{36}$$

where

$$\delta L \doteq [b(x, t)-\langle b(x, t)\rangle](\partial/\partial v) \tag{37}$$

However, this is inconsistent with the Fokker–Planck approximation even where the latter is expected to be valid. In the Fokker–Planck approximation,

$$\delta G = -G_0(\delta L)G \tag{38}$$

is used. In general, there is no reason to expect that (36) will agree with (38) once

$$\|G-G_0\| \ll \|G_0\| \tag{39}$$

no longer holds. Thus, they have assumed their conclusion.

5. RESULTS AND DISCUSSION

In this section, we study the three closure schemes discussed in the previous sections, using the results for $P(v, \tau; v')$ obtained analytically in Section 3. We separate the discussion into the cases of Gaussian statistics and ME statistics, and give, finally, a brief separate discussion of the short time results for both cases. In all of the figures, the velocity interval is given by $v' = 2.0$, $\Delta v = 10^{-2}$.

5.1. Gaussian Statistics

The result for the velocity space Green's function for Gaussian b is given by eq. (23). Then it follows that

$$-\frac{\partial}{\partial\tau}\int_{v'-\Delta v}^{v'+\Delta v} dv\, P(v, \tau; v') = \frac{1}{\pi^{1/2}\tau_c}\frac{\dot{\alpha}(\tau/\tau_c)}{\alpha(\tau/\tau_c)}\left(\frac{\Delta v}{b_0\tau_c}\right)\frac{1}{(4\alpha)^{1/2}} \times \exp\left[-\frac{1}{4\alpha}\left(\frac{\Delta v}{b_0\tau_c}\right)^2\right] \tag{40}$$

The right-hand side integrated with respect to v over the interval $(v_l, v_r) \doteq (v'-\Delta v, v'+\Delta v)$ can easily be obtained in an analytic form for the diffusion approximation (35a), and as expressions in which the v'' integration has been carried out but the τ'' integration still remains to be done (numerically) for the Bourret (35b) and direct-interaction (35c) approximations. The results are, respectively

$$\mathrm{FP_G} = \frac{1}{\pi^{1/2}\tau_c}\alpha^{-1}\left(\frac{\Delta v}{b_0\tau_c}\right)\frac{1}{(4\alpha)^{1/2}}\exp\left[-\frac{1}{4\alpha}\left(\frac{\Delta v}{b_0\tau_c}\right)^2\right] \tag{41a}$$

$$\mathrm{B_G} = \frac{1}{2\pi^{1/2}\tau_c}\left(\frac{\Delta v}{b_0\tau_c}\right)\int_0^{\tau/\tau_c} dy\, \alpha(y)^{-3/2} \times \exp\left[y-\frac{\tau}{\tau_c}-\left(\frac{\Delta v}{b_0\tau_c}\right)^2\frac{1}{4\alpha(y)}\right] \tag{41b}$$

$$\mathrm{DIA_G} = \frac{1}{2\pi^{1/2}\tau_c}\left(\frac{\Delta v}{b_0\tau_c}\right)\int_{\exp(-\tau/\tau_c)}^{1} dy\, y^{3/2}\left[y^2+\left(\frac{\tau}{\tau_c}-2\right)y+\exp(-\tau/\tau_c)\right]^{-3/2}$$
$$\times\exp\left[-\frac{1}{4}\left(\frac{\Delta v}{b_0\tau_c}\right)^2\frac{y}{y^2+(\tau/\tau_c)\,y+\exp(-\tau/\tau_c)}\right] \tag{41c}$$

In Figs. 1, 2, and 3 we have plotted the expressions (40)–(41c) as functions of τ for three sets of parameter values. The values used in Fig. 1 correspond to $\bar{K}(\tau) \ll 1$ over all times except a brief initial period during which the evolution of (40)–(41c) has hardly begun [i.e., all particles are still in the velocity (v_l, v_r)]. The agreement between the right-hand sides (41a–c) and the left-hand side (40) is very good. In Fig. 2, $\bar{K} = (0.1)^{1/2}$ at $t = 1.0$, beyond which the agreement is again good. In the short time regime where $\bar{K} \gtrsim 1$, the DIA agrees better than the other two closures. In Fig. 3, this trend is more markedly emphasized.

The increased departure between the left- and right-hand sides as b_0 is increased can be characterized by $\bar{K}_0 \doteq \bar{K}(\tau_{ev})$, the Kubo number (33) evaluated with τ equal to the characteristic time for the evolution of (40), say. A sensible definition of $\bar{K}_0$ must satisfy the criterion that the agreement

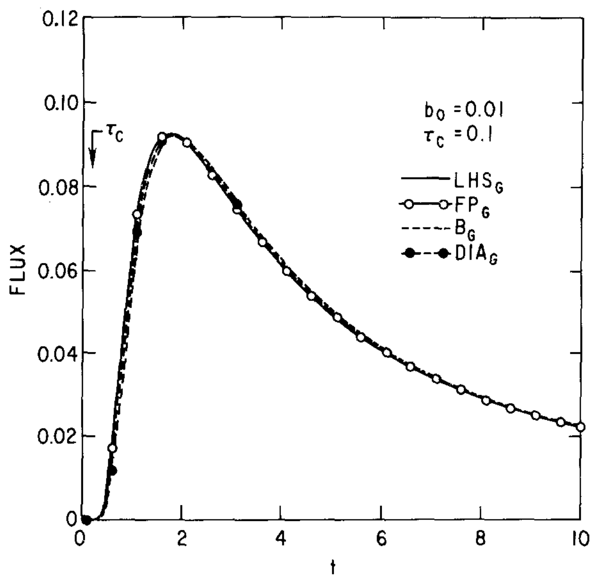

Fig. 1. Equations (40) and (41) for $b_0 = 0.01$, $\tau_c = 0.1$; this corresponds to $\bar{K}_0 \ll 1$ for most of the evolution.

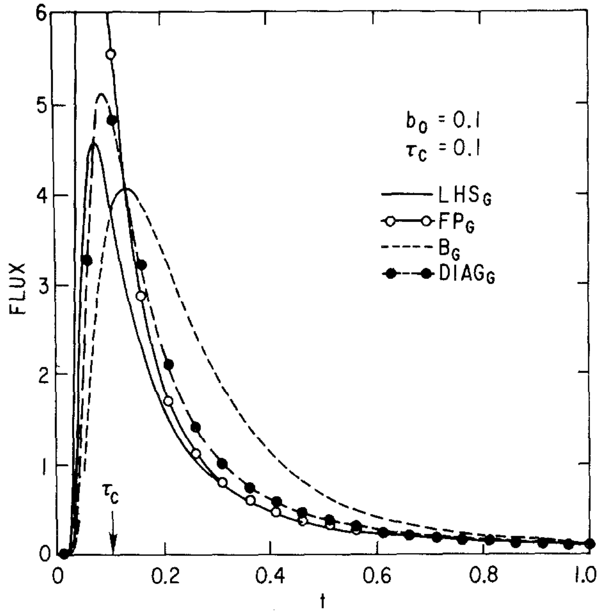

Fig. 2. Equations (40) and (41) for $b_0 = 0.1$, $\tau_c = 0.1$; this typically corresponds to $\bar{K}_0 \approx 1$.

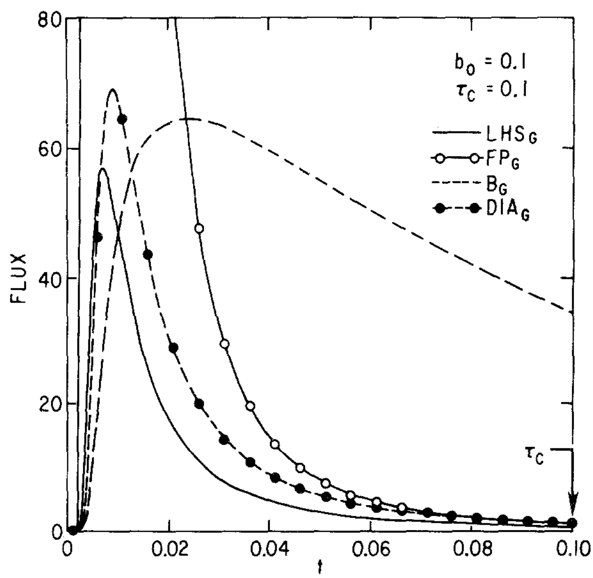

Fig. 3. Equations (40) and (41) for $b_0 = 1.0$, $\tau_c = 0.1$; this corresponds to $\bar{K}_0 \gg 1$ for most of the evolution.

between the right- and left-hand sides of the closure equations be good if and only if $\bar{K}_0 \ll 1$. Now, suppose the diffusion approximation is valid. Then $\alpha(\tau/\tau_c) \approx \tau/\tau_c$ and the effective evolution time for (40) is approximately given by

$$\tau_{ev} \doteq \left(\frac{\Delta v}{b_0 \tau_c}\right) \frac{\Delta v}{b_0} \tag{42}$$

so that

$$\bar{K}_0 = [\pi\alpha(\tau_{ev}/\tau_c)]^{-1/2} \approx \bar{K}_1 \doteq b_0 \tau_c/\Delta v \ll 1 \tag{43}$$

Conversely, if $\bar{K}_1 \ll 1$, then the exponential in (40) dominates the evolution until $\alpha \sim \bar{K}_1^{-2} \gg 1$, so that during most of the evolution of (40) the diffusion approximation is valid. Thus $\bar{K}_1 \ll 1$ is necessary and sufficient for the validity of the diffusion approximation over most of the time that the evolution of (40) occurs, so that $\bar{K}_1$ is also a sensible definition of the Kubo number. This argument is supported by the results displayed in Figs. 1–3, for which we have $\bar{K}_1$ respectively equal to 0.1, 1.0, and 10.0.

The slower decay of the value of the Bourret expression can be understood intuitively by noting that at any given time τ, contributions from all earlier times y, at which $\alpha^{-3/2}(y)$ is much larger, are only weakly damped by the $\exp[-(\tau/\tau_c - y)]$ factor in the integrand. In the Fokker–Planck expression, those contributions are absent; in the case of the DIA they are damped by the $G(x'', v, \tau; v'')$ factor in (35c) [which is obscured in the final form (41c) used for the numerical evaluation].

Summarizing, for b an Ornstein–Uhlenbeck process, all three closures are seen to work for small effective Kubo number, while for large effective Kubo number, none of the three closures work well quantitatively, although the right-hand side for only the DIA retains the qualitative features of the left-hand side.

5.2. Maasjost-Elsässer Statistics

The velocity space Green's function for the ME acceleration field is given by (29). The left-hand side of the closure equations integrated with respect to v over the interval $(v' - \Delta v, v' + \Delta v)$ can be evaluated to be

$$-\frac{\partial}{\partial \tau} \int_{v_l}^{v_r} dv\, P(v, \tau; v') = \frac{1}{\pi \tau_c} \frac{\dot{\alpha}(\tau/\tau_c)}{\alpha(\tau/\tau_c)} \left(\frac{\Delta v}{b_0 \tau_c}\right) \frac{1}{2\alpha(\tau/\tau_c)} K_0 \left[\left(\frac{\Delta v}{b_0 \tau_c}\right) (2\alpha)^{-1/2}\right] \tag{44}$$

The corresponding right-hand sides, which are displayed in Figs. 4–6, are

$$\mathrm{FP}_{\mathrm{ME}}=\frac{1}{\pi\alpha\tau_c}K_1\left[\frac{\Delta v}{b_0\tau_c(2\alpha)^{1/2}}\right] \tag{45a}$$

$$\mathrm{B}_{\mathrm{ME}}=\frac{1}{\pi\tau_c}\int_0^{\tau/\tau_c}dy\,\exp\left[-\left(\frac{\tau}{\tau_c}\right)-y\right]\frac{1}{\alpha(y)}K_1\left\{\left(\frac{\Delta v}{b_0\tau_c}\right)[2\alpha(y)]^{-1/2}\right\} \tag{45b}$$

$$\begin{aligned}\mathrm{DIA}_{\mathrm{ME}}=&\frac{1}{\pi\tau_c}\int_0^{\tau/\tau_c}dy\,\exp\left[-\left(\frac{\tau}{\tau_c}-y\right)\right]\\&\times(\alpha_1\alpha_2)^{-1/2}\exp\left[-\frac{1}{2}\left(\frac{\Delta v}{b_0\tau_c}\right)\left(\frac{1}{(2\alpha_1)^{1/2}}+\frac{1}{(2\alpha_2)^{1/2}}\right)\right]\\&\times\int_{-1}^{1}dx(1+\varepsilon x)(1-x^2)^{-1/2}[(2+\varepsilon x)^2-\varepsilon^2]\\&\times\exp\left[-\frac{1}{2}\left(\frac{\Delta v}{b_0\tau_c}\right)x\left(\frac{1}{(2\alpha_2)^{1/2}}-\frac{1}{(2\alpha_1)^{1/2}}\right)\right]\end{aligned} \tag{45c}$$

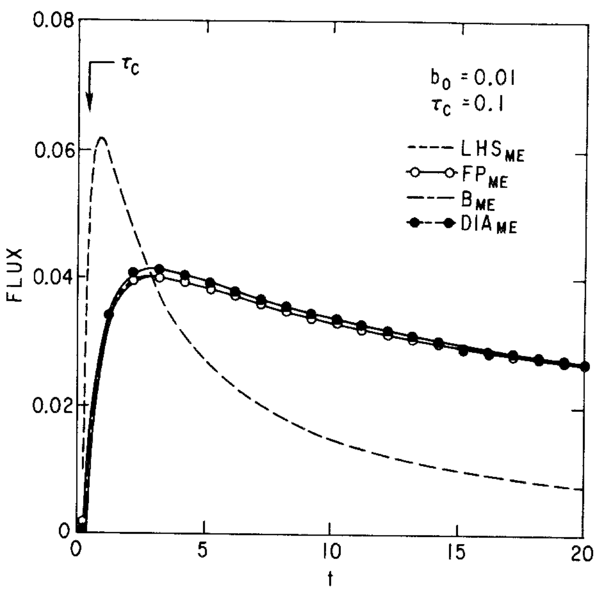

Fig. 4. Equations (44) and (45) for $b_0=0.01$, $\tau_c=0.1$.

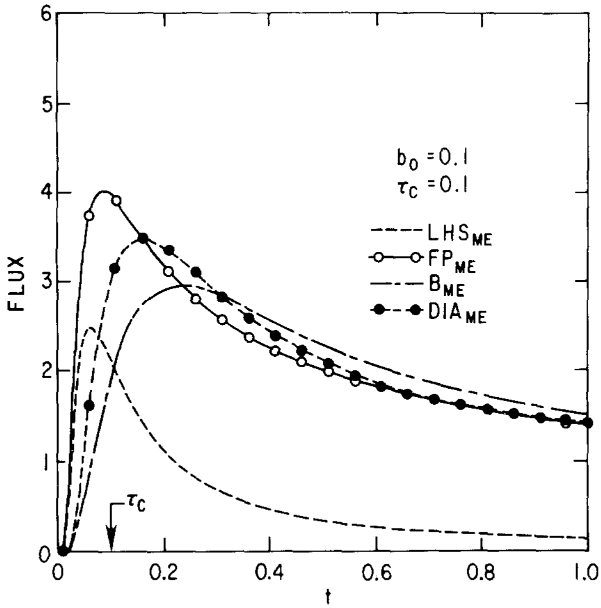

Fig. 5. Equations (44) and (45) for $b_0 = 0.1$, $\tau_c = 0.1$.

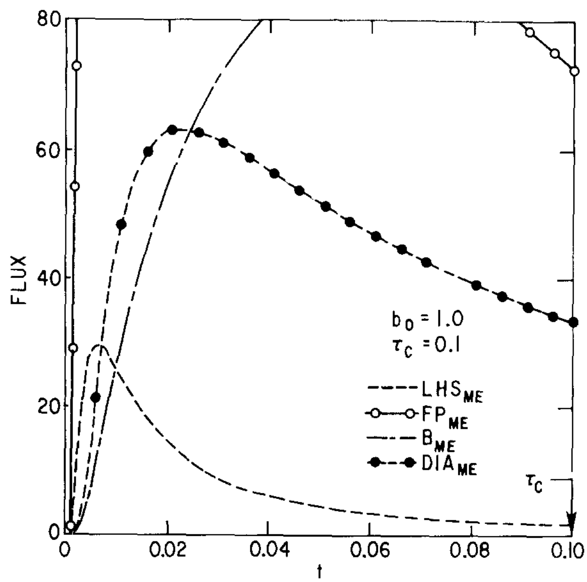

Fig. 6. Equations (44) and (45) for $b_0 = 1.0$, $\tau_c = 0.1$.

where

$$\alpha_1 \doteq \alpha(y) \tag{46a}$$

$$\alpha_2 \doteq \alpha[(\tau/\tau_c) - y] \tag{46b}$$

$$\varepsilon \doteq (\alpha_1^{1/2} - \alpha_2^{1/2})/(\alpha_1^{1/2} + \alpha_2^{1/2}) \tag{46c}$$

Equation (45c) has been obtained by turning the convolution integral with respect to v'' in (35c) into the inverse Fourier transform of the product of two Fourier-transformed Green's functions, then closing the inversion contour around a finite branch cut in the upper half plane. The x integration can then be handled by using the identity

$$\int_{-1}^{1} dx(1-x^2)^{-1/2} f(x) = 2\int_0^1 dz[f(z^2-1) + f(1-z^2)] \tag{47}$$

for any function $f(x)$ bounded on $(-1, 1)$.

In none of Figs. 4–6 is there close agreement between the left-hand side (44) and any of the right-hand sides (45a)–(45c). This is in agreement with the Kubo number criterion with the Kubo number given by (32), which is ∞ in the case of the ME process, irrespective of the values of b_0, τ_0, or Δv.

5.3. Short-Time Results

Maasjost and Elsässer found that [when integrated with respect to v over $(v' - \Delta v, v' + \Delta v)$ and differentiated with respect to τ] the approximate solution valid for $\tau \ll \tau_c$ when extrapolated into the middle and long-time regimes gave "excellent agreement with the numerical results in the middle and final stages of the interaction." Since the long-time regime for ME statistics is, strictly, inaccessible to our analysis, we cannot check this claim. However, for the cases studied here, embodied in Eqs. (40) and (44), we find that the long-time behavior is not correctly given by the short-time results extrapolated to long times. The long-time asymptotic behaviors are

$$\frac{1}{2\pi^{1/2}}\left(\frac{\Delta v}{b_0\tau_c}\right)\tau^{-1}\left(\frac{\tau_c}{\tau}\right)^{1/2}\left(1 + O\left\{\frac{\tau_c}{\tau}\left[1 + \left(\frac{\Delta v}{b_0\tau_c}\right)^2\right]\right\}\right)$$
$$\sim \tau^{-3/2} \quad \left[\tau \gg \left(\frac{\Delta v}{b_0\tau_c}\right)^2 \tau_c\right] \tag{48}$$

for Gaussian statistics, and

$$\frac{1}{2^{3/2}\pi}\left(\frac{\Delta v}{b_0\tau_c}\right)\tau^{-1}\left(\frac{\tau_c}{\tau}\right)^{1/2}\left[\ln\left(\frac{\tau}{\tau_c}\right) + O(1)\right] \sim \tau^{-3/2}\ln(\tau) \quad \left[\tau \gg \left(\frac{\Delta v}{b_0\tau_c}\right)^2 \tau_c\right] \tag{49}$$

for ME statistics. The short-time results extrapolated to long times in the time asymptotic limit $\tau \gg \Delta v/b_0$ are

$$\left(\frac{2}{\pi}\right)\left(\frac{\Delta v}{b_0 \tau}\right)\tau^{-1}\left\{1+O\left[\left(\frac{\Delta v}{b_0 \tau}\right)^2\right]\right\} \sim \tau^{-2} \tag{50}$$

for Gaussian statistics, and

$$\left(\frac{2}{\pi\tau}\right)\left(\frac{\Delta v}{b_0 \tau}\right)\left[\ln\left(\frac{b_0 \tau}{\Delta v}\right)+O(1)\right] \sim \tau^{-2}\ln(\tau) \tag{51}$$

for ME statistics. Thus, there is a disagreement between (48) and (50) and between (49) and (51), the latter in each pair having an extra factor of $\tau^{-1/2}$. Figures 7 and 8 show the extrapolated short time and the exact solutions both for Gaussian and ME statistics. In Fig. 7, corresponding to Fig. 6 of ME, the short-time solution follows the exact solution for ME statistics. In Fig. 8 we have reduced τ_c by a factor of 10. This increases the evolution time scale and hence changes the characteristic value of τ/τ_c, thus separating the short-time and exact solutions. Thus, we find no systematic agreement between the exact solutions and the short-time solutions extrapolated to long times.

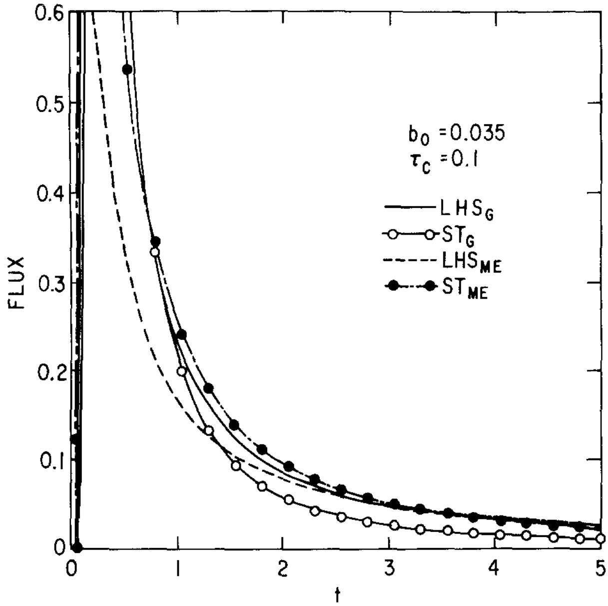

Fig. 7. Equations (40) and (44) and their short-time asymptotic forms denoted, respectively, by ST_G and ST_{ME} for $b_0 = 3.5 \times 10^{-2}$ and $\tau_c = 1/9$.

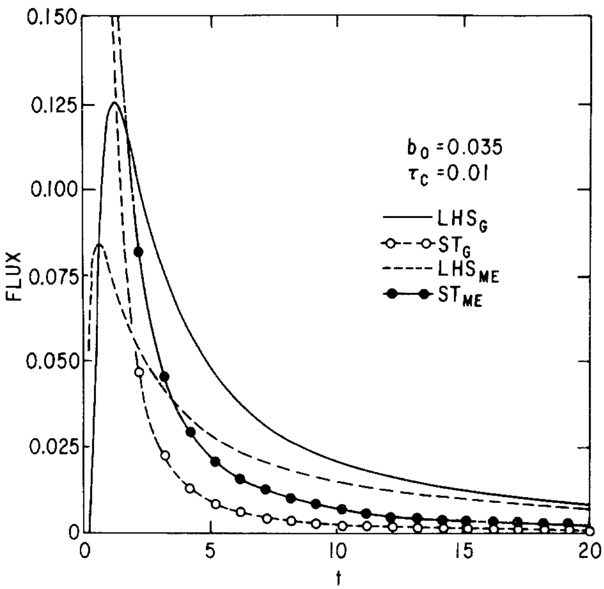

Fig. 8. Equations (40) and (44) and their short-time asymptotic forms for $b_0 = 3.5 \times 10^{-2}$, $\tau_c = 1/90$.

6. CONCLUSIONS

We have studied in analytically tractable cases the acceleration of a particle in a stochastic acceleration field given in one case by an Ornstein–Uhlenbeck process and in the other by a process used by ME in their numerical experiments. Following ME, the results were used to test the accuracy of the Fokker–Planck, the Bourret, and the direct-interaction approximation, as well as statements made on this subject by ME. For the Gaussian process, the effective Kubo number is finite and may be taken to be $\bar{K} = b_0 \tau_c / \Delta v$. For small $\bar{K}$, all three closures agree with each other and with the analytical solution. For $\bar{K} \gtrsim 1$, none of the three closures show good quantitative agreement with the left-hand side although the DIA does far better than the other two. The DIA is the only one for which the right-hand side is qualitatively correct, contrary to the conclusion of ME. For the ME process, $\bar{K} = \infty$ and all three closures fail. This is in agreement with the fact that in the derivation of all three closures, the acceleration is assumed to be near-Gaussian, which is not the case for the ME process, a point which was not stressed by ME. Finally, we found no systematic agreement between the exact solutions and the short-time solutions extrapolated to long times for either process. We note that for none of the sets of parameter values displayed does the DIA agree well quantitatively over the whole time domain if the other two closures disagree somewhere. Clearly, qualitative agreement of the expressions used in this test does not necessarily imply qualitative agreement of the solutions of the closure equations, nor does the reverse implication necessarily hold. As a referee has stressed, closures such as the DIA, which are realizable in the sense of possessing an underlying stochastic model,[13,14] have certain self-regulatory properties which may make the fully self-consistent solution of the closure better-behaved than the test used here might indicate. Thus, from the present test we cannot definitively conclude whether or not the DIA is useful in connection with the stochastic acceleration problem, although we have shown that it does behave differently from the other two closures and that the reasons given by ME for rejecting it are unjustified. We should also emphasize that the choices of one dimension, $v' = 2.0$, $\Delta v \ll v'$, and the analysis based on the spatially independent acceleration were made specifically in order to cover the parameter ranges studied by ME. (See Appendix A for the generalization to spatially varying fields.) For parameter ranges not covered by Appendix A, provided trapping is unimportant, it is possible that the differences between the Fokker–Planck and Bourret approximations, on the one hand, and the DIA, on the other, become even more marked than for the examples studied in this paper.

ACKNOWLEDGMENT

We wish to thank both referees for their helpful, pertinent remarks, which we have incorporated into the paper.

APPENDIX A. EXTENSION OF THE RESULTS OF SECTION 2 TO SPATIALLY DEPENDENT GAUSSIAN FORCE FIELDS

The integration of eqs. (8) gives explicit integral equations which can be iterated to yield expansions of x and v as functional power series in b. The result for v_0 up to second order is

$$\begin{aligned} v_0 \approx v &- \int_0^t dt'\, b[x - v(t-t')] \\ &- \int_0^t dt' \int_{t'}^t dt''(t''-t')\, b[x - v(t-t''), t''] \frac{\partial}{\partial x} b[x - v(t-t'), t'] \end{aligned} \tag{A.1}$$

Upon keeping terms in (A.1) up to first order in b, inserting the result into (9), and using the Fourier representation of the δ function, we obtain

$$P(v, t; v') \approx \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp[ik(v-v')] \left\langle \exp\left\{-ik \int_0^t dt'\, b[x - v(t-t'), t')]\right\}\right\rangle \tag{A.2}$$

For the Gaussian field with

$$\langle b(x, t)\rangle = 0 \tag{A.3a}$$

and

$$\begin{aligned} C(x_1, t_1; x_2, t_2) &\doteq \langle b(x_1, t_1)\, b(x_2, t_2)\rangle \\ &= b_0^2 \exp(-|x_1 - x_2|/l_c - |t_1 - t_2|/\tau_c) \end{aligned} \tag{A.3b}$$

eq. (A.2) gives the same result as for the $l_c = \infty$ case with the replacement $\tau_c \to \tau_{ac}$, where

$$\tau_{ac} \doteq \left(\frac{1}{\tau_c} + \frac{v}{l_c}\right)^{-1} \tag{A.4}$$

is the effective autocorrelation time for (A.2). The statements of Section 2.2 carry over exactly as given there except that eq. (13) is replaced by

$$\hat{b}(\eta) = b(v\tau_{ac}\eta, \tau_{ac}\eta)/b_0 \tag{A.5}$$

To estimate the range of validity of the results of Section 5 with τ_c replaced by τ_{ac}, we need to consider two sources of error. The first is the neglect of terms of second and higher order with respect to b. The second is that, since τ_{ac} depends on v, the integration of the expressions resulting from (A.2) with respect to v' can no longer be carried out exactly. To estimate the effect of the second-order term in (A.1), we can note that this term, when retained in the calculation of the drag coefficient, causes the v-dependent diffusion coefficient to appear between, rather than inside, the v derivatives.[1,6] Consider the addition to the diffusion equation of a small drag term with coefficient δ:

$$\frac{\partial g_1}{\partial t} = D \frac{\partial^2}{\partial v^2} g_1 + \delta \frac{\partial}{\partial v} g_1 \tag{A.6}$$

The solution to this equation which initially satisfies

$$g_1(v, 0; v') = \delta(v - v')$$

is

$$g_1(v, t; v') = (4\pi Dt)^{-1/2} \exp(-(v - v' + \delta t)^2/4Dt) \tag{A.7}$$

In order to neglect δ, we thus require

$$|v - v'|\, \delta/D \ll 1 \tag{A.8}$$

and

$$\delta^2 t/D \ll 1 \tag{A.9}$$

For the purposes of estimation, we take

$$D = b_0^2 \bar{\tau}_{ac}$$

where

$$\frac{1}{\bar{\tau}_{ac}} \doteq \frac{v'}{l_c} + \frac{1}{\tau_c}$$

and

$$\delta = \frac{\partial D}{\partial v'}$$

Upon inserting this into (A.8) and (A.9), respectively, we obtain

$$\frac{|v - v'|\, \tau_{ac}}{l_c} \ll 1 \quad \text{or} \quad v' + \frac{l_c}{\tau_c} \gg |v - v'| \tag{A.10}$$

and

$$\frac{b_0^2\, \tau_{ac}^3\, t}{l_c^2} \ll 1 \tag{A.11}$$

Now we can associate with (A.6) a Langevin-type equation of the form

$$(dv/dt) + \delta v = F(t) \tag{A.12}$$

where $F(t)$ is a random force satisfying

$$D = \int_0^\infty dt \langle F(t)\, F(0) \rangle \tag{A.13}$$

Physically, the conditions (A.11) then represent the necessary and sufficient criteria for the drag, which causes a displacement of the entire velocity profile, to be negligible.

Alternatively, we can express the criteria (A.10) and (A.11) in terms of the characteristic times for a particle with initial velocity v' to leave the interval $(v' - \Delta v,\ v' + \Delta v)$, where $\Delta v \doteq |v - v'|$, due to drag and diffusion, respectively:

$$\tau_\delta \doteq \frac{\Delta v}{\delta} \tag{A.14a}$$

$$\tau_D \doteq \frac{(\Delta v)^2}{D} \tag{A.14b}$$

Equations (A.10) and (A.11) then become

$$\frac{\tau_D}{\tau_\delta} \ll 1 \tag{A.15a}$$

$$\frac{(\delta t)^2}{Dt} \ll 1 \tag{A.15b}$$

The first of these says that the particle must leave more quickly by diffusion than by drag, while the second says that the velocity change due to drag must be smaller than that due to diffusion.

A direct mathematical estimate of the criterion for the validity of (A.2) can be made by rewriting (A.1) in the form

$$-ikv_0 = -ikv + \int d\xi_1\, A(\xi_1)\, b(\xi_1) - \frac{1}{2}\int d\xi_1\, d\xi_2\, B(\xi_1, \xi_2)\, b(\xi_1)\, b(\xi_2) \tag{A.16}$$

where

$$\xi_i \doteq (x_i, t_i) \tag{A.17a}$$

$$A(\xi_i) \doteq ik\chi_{(0,t)}(t_i)\, \delta\{x_i - [x - v(t - t_i)]\} \tag{A.17b}$$

$$\begin{aligned} B(\xi_1, \xi_2) \doteq {} & -2ik\chi_{(0,t)}(t_1)\, \chi_{(t_1,t)}(t_2)(t_1 - t_2) \\ & \times \delta\{x_1 - [x - v(t - t_1)]\}\, \delta\{x_2 - [x - v(t - t_2)]\} \end{aligned} \tag{A.17c}$$

Upon performing the ensemble average, but now keeping terms up to second order, we have formally

$$\begin{aligned} \langle \exp(-ikv_0) \rangle \approx {} & \exp(-ikv)[\det(1 + CB)]^{-1/2} \\ & \times \exp\left\{\frac{1}{2}\int d\xi_1\, d\xi_2\, A(\xi_1)\, A(\xi_2)(C^{-1} + B)^{-1}(\xi_1, \xi_2)\right\} \end{aligned} \tag{A.18}$$

where

$$C(\xi_1, \xi_2) \doteq \langle b(\xi_1)\, b(\xi_2) \rangle$$

and $\langle b(\xi) \rangle = 0$ has been assumed.

We wish to estimate the effect of B. Consider an integral of the form

$$
\begin{aligned}
I &\doteq \int \frac{dk}{2\pi} \left\langle \exp(ik\alpha b - \frac{1}{2} k\beta b^2) \right\rangle \exp(ik\,\Delta v) \\
&= \int \frac{dk}{2\pi} (1 + k\beta b_0^2)^{-1/2} \exp\left(- \frac{k^2 b_0^2 \alpha^2}{2(1 + k\beta b_0^2)} \right) \exp(-ik\,\Delta v) \\
&= \frac{1}{(2\pi)^{1/2} b_0 \alpha} \left[1 + O\left(\frac{\beta\,\Delta v}{\alpha^2}\right) + O\left(\frac{\beta^2 b_0^2}{\alpha^2}\right) \right] \\
&\quad \times \exp\left(- \frac{1}{2} \frac{(\Delta v)^2}{b_0^2 \alpha^2} \left[1 + O\left(\frac{\beta\,\Delta v}{\alpha^2}\right) \right] \right)
\end{aligned} \tag{A.19}
$$

where α and β are small quantities and $\beta > 0$. We see from (A.19) that β can be neglected if and only if all of the following hold

$$\frac{\beta\,\Delta v}{\alpha^2} \ll 1 \tag{A.20a}$$

$$\frac{\beta^2 b_0^2}{\alpha^2} \ll 1 \tag{A.20b}$$

$$\frac{(\Delta v)^3 \beta}{b_0^2 \alpha^4} \ll 1 \tag{A.20c}$$

By comparing (A.18) and (A.19), we see that necessary conditions for the neglect of B in (A.18) can be obtained from the results (A.20) by setting

$$
\begin{aligned}
\beta b_0^2 \rightarrow \frac{ACBCA}{kACA} &= O[k^{-1}\,\mathrm{Tr}(CB)] \\
&= O(b_0^2 \tau_{ac}^2 t/l_c) \qquad (t \gg \tau_{ac}) \\
&= O(b_0^2 t^3/l_c) \qquad (t \ll \tau_{ac})
\end{aligned} \tag{A.21a}
$$

$$
\begin{aligned}
\alpha^2 b_0^2 \rightarrow k^{-2}\,\mathrm{tr}(ACA) &= O(b_0^2 \tau_{ac} t) \qquad (t \gg \tau_{ac}) \\
&= O(b_0^2 t^2) \qquad (t \ll \tau_{ac})
\end{aligned} \tag{A.21b}
$$

Upon making these insertions, (A.20a) and (A.20b) result, respectively, in (A.10) and (A.11), while (A.20c) gives

$$(\Delta v)^3/b_0^2 l_c t \ll 1 \quad \text{or} \quad t \gg t_0 \doteq (\Delta v)^3/b_0^2 l_c \tag{A.22}$$

Equation (A.22) is not obtained by the analysis proceeding from (A.6) since it corresponds to the effect of terms containing derivatives with respect to v higher than second.

Assessing the numerical values of these criteria for Figs. 1–3, we find that (A.10) and (A.11) are well-satisfied even for $l_c = 0$. If we take $\tau_{ac} = 0.1$ for Figs. 1–3 and $l_c = v\tau_c$, we obtain $t_0 = 10^{-3}$, 10^{-5}, and 10^{-7} for Figs. 1–3, respectively.

If we could show that the terms of higher order in β in (A.19) are negligible when the appropriate operator replacements are made (for example, if $\mathrm{tr}[AC(BC)^n A] = O\{\mathrm{tr}(ACA)[\mathrm{tr}(BC)]^n\}$), then the conditions (A.8), (A.9), and (A.22) would also be sufficient. This seems plausible, although we have been unable to prove it.

The second source of error, namely the v dependence of τ_{ac}, causes an error in the results of Section 5 for the Fokker–Planck and Bourret approximations, which is small provided (A.8) holds. In the case of the DIA expression (41c), it is necessary that the spatial length scale of G be much smaller than l_c. This gives the necessary condition

$$t \ll \left(\frac{l_c}{b_0 \tau_{ac}^{1/2}} \right)^{2/3} \tag{A.23}$$

For $\tau_c = \infty$, this condition gives $t \ll 16$, 3.4, and 0.74, respectively, for Figs. 1–3, which is more stringent than (A.9).

An analysis of the type presented in this appendix is expected to hold even for time-independent acceleration fields provided that particle trapping and large deflections—for example, from single peaks of a potential acceleration field—are unimportant. This is not the case for the one-dimensional time-independent problem.

APPENDIX B. SOME PROPERTIES OF THE MAASJOST AND ELSÄSSER FIELD

First we derive the characteristic functional (27). By definition, we have

$$G[k] \doteq \left\langle \exp\left[-i \int dx\, dt\, k(x, t)\, b_x(x)\, b_t(t) \right] \right\rangle \tag{B.1}$$

Since b_x and b_t are both Gaussian processes, we can write formally

$$G[k] = [\det(2\pi\sigma_x) \det(2\pi\sigma_t)]^{-1/2} \int d[b_x]\, d[b_t]$$
$$\times \exp\left(-\frac{1}{2} b_x \cdot \sigma_x^{-1} \cdot b_x - \frac{1}{2} b_t \cdot \sigma_t^{-1} \cdot b_t - i b_x \cdot k \cdot b_t \right)$$

where

$$b_\alpha \cdot f \doteq \int d\alpha\, b_\alpha(\alpha) f(\alpha) \qquad (\alpha = x \text{ or } t)$$

and

$$\sigma_a(\alpha_1, \alpha_2) \doteq \langle b_\alpha(\alpha_1)\, b_\alpha(\alpha_2) \rangle \tag{B.2}$$

Upon performing the functional integration over $[b_t]$, we find

$$G[k] = [\det(2\pi\sigma_x)]^{-1/2} \int d[b_x] \exp\left[-\frac{1}{2} b_x \cdot (\sigma_x^{-1} + k \cdot \sigma_t \cdot k^T) \cdot b_x \right] \tag{B.3}$$

where

$$k^T(t, x) \doteq k(x, t)$$

Then, upon performing the integration over $[b_x]$ we obtain

$$G[k] = [\det(I_x + \sigma_x \cdot k \cdot \sigma_t \cdot k^T)]^{-1/2} \tag{B.4}$$

which, with the matrix products written out explicitly and the correlation functions from (25) inserted, gives (27).

It is possible to make some observations regarding (A.2) for this acceleration field, although the justification given in Appendix A of it as an approximation to $D(v, \tau; v')$ is no longer valid. The term in the ensemble average brackets in (A.2) is just $G[k]$ for the test function

$$k(x, t) = iA(x, t) \tag{B.5}$$

where $A(x, t)$ is given by (A.17b). It can again be written as the solution of the stochastic oscillator problem as in Section 2 and with (A.5) written explicitly as

$$\hat{b}(\eta) = \hat{b}_x(\eta)\, \hat{b}_t(\eta) \tag{B.6}$$

where

$$\hat{b}_x(\eta) \doteq \sigma_x^{-1} b_x(v\tau_c\eta)$$
$$\hat{b}_t(\eta) \doteq \sigma_y^{-1} b_t(\tau_c\eta)$$

From (25) it follows that

$$\langle \hat{b}_x(\eta_1)\, \hat{b}_x(\eta_2) \rangle = \exp(-|\eta_1 - \eta_2|/T_x) \tag{B.7a}$$
$$\langle \hat{b}_t(\eta_1)\, \hat{b}_t(\eta_2) \rangle = \exp(-|\eta_1 - \eta_2|/T_t) \tag{B.7b}$$

where

$$T_x \doteq l_c/(v\tau_{ac}), \qquad T_t \doteq \tau_c/\tau_{ac}$$

The stochastic oscillator solution is formally

$$R_K(\tau) = [\det(I + K^2 \hat{C}_x \hat{C}_t)]^{-1/2} \tag{B.8}$$

where

$$\hat{C}_\alpha(\eta_1, \eta_2) \doteq \chi_{(0, t/\tau_{ac})}(\eta_1) \langle \hat{b}_\alpha(\eta_1)\, \hat{b}_\alpha(\eta_2) \rangle$$

(B.8) can be substituted into (17) to give the (still formal) solution. It is difficult to make much further progress except in the case

$$t/\tau_{ac} \ll T_x \quad \text{or} \quad T_t \tag{B.9}$$

in which limit (28) and (29) are valid. A series expansion of (B.8) in K is of no use since large values of K contribute in (17). If

$$T_x = T_t = 2 \tag{B.10}$$

then it is possible to obtain the eigenvalue condition which gives the eigenvalues whose product makes up (B.8). The result is

$$R_K(\tau) = \prod_i (1 + K^2 \lambda_i)^{-n_i/2} \tag{B.11}$$

where n_i is the degeneracy of the ith eigenvalue, $\lambda_i \doteq [4/(\alpha_i^2 + 1)]^2$, and α_i satisfies $\alpha = \tan(\alpha\tau/4)$ or $\alpha = \cot(\alpha\tau/4)$. Even though the eigenvalues can each be computed with arbitrary precision, we have only been able to use them to cary out an asymptotic evaluation of (17) in the short-time limit which, given (B.10), is equivalent to the case (B.9), and which can easily be obtained without recourse to the above scheme or any closure approximations. (See, for example, ME.)

An argument similar to that leading to (A.20) for the Gaussian field can be applied. Consider an integral of the form

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(ik\,\Delta v)(2\pi\sigma_x\sigma_t)^{-1} \\ &\quad \times \int db_x\, db_t \exp\left(ik\alpha b_x b_t - \frac{1}{2} k\beta b_x^2 b_t^2 - \frac{b_x^2}{2\sigma_x^2} - \frac{b_t^2}{2\sigma_t^2}\right) \\ &= \frac{1}{\pi\alpha b_0} \left\{ K_0\left(\frac{\Delta v}{\alpha b_0}\right) + \frac{1}{2} i \left(\frac{\Delta v \beta}{\alpha^2}\right) \left[\left(\frac{\Delta v}{\alpha b_0}\right) K_1\left(\frac{\Delta v}{\alpha b_0}\right) - K_0\left(\frac{\Delta v}{\alpha b_0}\right) \right] + O(\beta^2) \right\} \\ &\qquad (\beta \to 0) \end{aligned} \tag{B.12}$$

Terms involving nonzero powers of β can be neglected if and only if (A.20a) holds. Evaluating P as given by (A.2) but for the ME field, keeping the second-order term in b, and expanding to first order in that term, we find that provided one eigenvalue of $\sigma_x \cdot A \cdot \sigma_t \cdot A$ dominates over the others [which is true, for example, if (B.9) holds], then we can make the replacements (A.21), which yield (A.10). Again, if the higher terms in the operator expansions corresponding to an expansion in β are also well-behaved, then (A.10) and (B.9) are necessary and sufficient conditions for the results of Section 3 for ME statistics to be extended to the finite l_c case by applying (A.4).

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